WEAK EXTENT, SUBMETRIZABILITY AND DIAGONAL DEGREES

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ABSTRACT. We show that if X has a zero-set diagonal and X^2 has countable weak extent, then X is submetrizable. This generalizes earlier results from Martin and Buzyakova. Furthermore we show that if X has a regular G_δ-diagonal and X^2 has countable weak extent, then X condenses onto a second countable Hausdorff space. We also prove several cardinality bounds involving various types of diagonal degree.

1. INTRODUCTION

A space is called submetrizable if it admits a coarser metrizable topology. The diagonal of X^2, denoted by Δ_X, is the set $\{(x,x) : x \in X\}$. A space X is said to have a zero-set diagonal if there is a continuous function $f : X^2 \to [0,1]$ such that $\Delta_X = f^{-1}(0)$ and X is said to have a regular G_δ-diagonal if Δ_X is a regular G_δ-subset of X, i.e. it is the intersection of countably many closed neighbourhoods.

It is well-known that every submetrizable space has a zero-set diagonal, but the converse is false in general (see the example constructed in [15] and the remarks on it made in [2, Example 2.17]). This suggests to find conditions for a space with a zero-set diagonal to be submetrizable.

For example, in [13] H.W. Martin proved that separable spaces having a zero-set diagonal are submetrizable. In another direction, in [7] R.Z. Buzyakova showed that if X has a zero-set diagonal and X^2 has countable extent then X is submetrizable. Separability and countable extent are independent properties, but they have a quite natural common weakening, namely countable weak extent. In the first part of our paper, we give a simultaneous generalization of both the previous results by showing that spaces having a zero-set diagonal and whose square has countable weak extent are submetrizable.

Buzyakova also proved (see [7, Theorem 2.4 & 2.5]) that if X has a regular G_δ-diagonal and either it is separable or X^2 has countable extent, then X condenses onto a second-countable Hausdorff space. Again, we give a simultaneous generalization of both these results by showing that if X^2 has countable weak extent and a regular G_δ-diagonal, then X condenses onto a second-countable Hausdorff space.

In the second part of the paper we will study cardinality bounds on a space X according to the specific way its diagonal is embedded in X^2.

Date: December 6, 2011.

2000 *Mathematics Subject Classification.* 54A25, 54C10, 54D20, 54E99.

Key words and phrases. Submetrizable spaces, weak extent, regular G_δ-diagonal, rank n-diagonal, weak Lindelöf number.

2. Notation and terminology

For all undefined notions we refer to [10].

Recall that X condenses onto Y if there is a continuous bijection from X onto Y. So a space is submetrizable if and only if it condenses onto a metrizable space. The extent of a space X, denoted by $e(X)$, is the supremum of the cardinalities of closed and discrete subsets of X. The weak extent of a space X, denoted by $we(X)$, is the least cardinal number κ such that for every open cover $\mathcal{U}$ of X there is a subset A of X of cardinality no greater than κ such that $\mathrm{St}(A,\ \mathcal{U}) = X$. It is clear that $we(X) \le d(X)$ and $we(X) \le e(X)$. Note that spaces with countable weak extent are called star countable by several authors (see, for instance [1]). For a space X the weak-Lindelöf number of X, denoted by $wL(X)$, is the least cardinal κ such that every open cover of X has a subfamily of cardinality no greater than κ whose union is dense in X.

Whenever $\mathcal{B}$ is a collection of subsets of X and $A \subseteq X$, the star at A with respect to $\mathcal{B}$, denoted by $\mathrm{St}(A,\ \mathcal{B})$, is defined by the formula

$$\mathrm{St}(A,\ \mathcal{B}) = \bigcup\{B \in \mathcal{B} : A \cap B \ne \emptyset\}.$$

If we let $\mathrm{St}^0(A,\ \mathcal{B}) = A$ then, for $n \in \omega$, the n-star around A is defined by induction:

$$\mathrm{St}^{n+1}(A,\ \mathcal{B}) = \mathrm{St}(\mathrm{St}^n(A,\ \mathcal{B}),\ \mathcal{B}).$$

Note that $\mathrm{St}^1(A,\ \mathcal{B}) = \mathrm{St}(A,\ \mathcal{B})$. If $A = \{a\}$ we write $\mathrm{St}^n(a,\ \mathcal{B})$ instead of $\mathrm{St}^n(A,\ \mathcal{B})$.

If $n \in \omega$, and κ is an infinite cardinal, we say that a space X has a rank n G_κ-diagonal (a strong rank n G_κ-diagonal) if there is a sequence $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ of open covers of X such that for all $x \ne y$, there is some $\alpha < \kappa$ such that $y \notin St^n(x,\mathcal{U}_\alpha)$ ($y \notin \overline{St^n(x,\mathcal{U}_\alpha)}$). When $\kappa = \omega$, we will simply write rank n-diagonal. We will denote the minimal cardinal κ such that X has a rank n G_κ-diagonal or a strong rank n G_κ-diagonal by $\Delta_n(X)$ and $s\Delta_n(X)$, respectively. The formula $\Delta_n(X) \le \min\{\Delta_{n+1}(X), s\Delta_n(X)\}$ is obviously true. If $n = 1$ we will omit the number 1.

Recall that a space has a G_δ-diagonal if and only if it has a rank 1-diagonal (this was proved by Ceder in [9, Lemma 5.4]). In analogy to Ceder's result, Zenor proved in [17, Theorem 1] that a space X has a regular G_δ-diagonal if and only if there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for all $x \ne y$, there is a neighbourhood U of x and some $n \in \omega$ such that $y \notin \overline{St(U,\mathcal{U}_n)}$.

In particular, if a space has a strong rank 2-diagonal, then it has a regular G_δ-diagonal. We must say that at present we do not know any example of spaces having a regular G_δ-diagonal that does not have a strong rank 2-diagonal. Even more intriguing is the relationship between regular G_δ-diagonal and rank 2-diagonal. It is well-known that there exists a space with a rank 2-diagonal that does not have a regular G_δ-diagonal, namely the Mrowka space Ψ (see [2]). This easily follows from a result of McArthur ([14]), stating that a pseudocompact space with a regular G_δ-diagonal is metrizable. But the following question from A. Bella ([4]) is still open:

Question 2.1. *Does any space with a regular G_δ-diagonal have a rank 2-diagonal?*

A good reason for asking such a question comes out from a comparison of the following two facts. In [4] Bella proved that a ccc space with a rank 2-diagonal

has cardinality not exceeding 2^ω. Much more recently and with a certain effort, in [8] Buzyakova has shown that a ccc space with a regular G_δ-diagonal has again cardinality not exceeding 2^ω. Therefore, a positive answer to the previous question would imply a trivial proof of the latter result from the former.

3. Zero-set diagonal vs submetrizability

The aim of this section is to provide a simultaneous generalization of Martin and Buzyakova's results. The obvious way to accomplish this is by using the weak extent. However, we actually present a formally stronger result obtained by means of an even weaker form of the weak extent of a square.

The weak double extent of a space X, denoted by $wee(X)$, is the smallest cardinal κ such that whenever $\mathcal{U}$ is an open cover of X^2, there exists some $A \subseteq X$ with $|A| \leq \kappa$ such that

$$\mathrm{St}(X \times A, \mathcal{U}) = X^2.$$

The following is obvious.

Proposition 3.1. *For any space X, we have $we(X) \leq wee(X) \leq we(X^2)$.*

By using Example 3.3.4 in [16], we are going to provide a space X such that $we(X) < wee(X)$. Let Ψ be the Mrowka space $\mathcal{A} \cup \omega$, where the cardinality of $\mathcal{A}$ is $\mathfrak{c}$, and let Y be the one-point compactification of a discrete space D of cardinality $\mathfrak{c}$. The space $X = \Psi \oplus Y$ is the topological sum of a separable space and a compact space and so we have $we(X) = \omega$. Write $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $D = \{d_\alpha : \alpha < \mathfrak{c}\}$. Let

$$\begin{aligned} U_1 &= \{\Psi \times \{d_\alpha\} : \alpha < \mathfrak{c}\}, \\ U_2 &= \{(\{A_\alpha\} \cup A_\alpha) \times (Y \setminus \{d_\alpha\}) : \alpha < \mathfrak{c}\}, \\ U_3 &= \{\{n\} \times Y : n < \omega\}, \end{aligned}$$

and finally $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \{Y \times Y\} \cup \{\Psi \times \Psi\} \cup \{Y \times \Psi\}$.

Of course the family $\mathcal{U}$ is an open cover of X^2. Assume that there exists a countable set $C \subseteq X$ such that $St(X \times C, \mathcal{U}) = X^2$. This in turn would imply the relation $St(\Psi \times (C \cap Y), \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3)) = \Psi \times Y$. Since we have $\Psi \times Y \setminus (\bigcup \mathcal{U}_2 \cup \bigcup \mathcal{U}_3) \supseteq \{(A_\alpha, d_\alpha) : \alpha < \mathfrak{c}\}$, it should be $\{(A_\alpha, d_\alpha) : \alpha < \mathfrak{c}\} \subseteq St(\Psi \times (C \cap Y), \mathcal{U}_1)$. But this would imply $D \subseteq C \cap Y$, which is a contradiction. This suffices for the proof that $wee(X) > \omega = we(X)$.

A further look shows that we actually have $wee(X) = \mathfrak{c}$. By repeating the same construction, with the Katetov's extension in place of Ψ and with D a set of cardinality $2^{\mathfrak{c}}$, we get a Hausdorff space X such that $we(X) = \omega$ and $wee(X) = 2^{\mathfrak{c}}$.

Right now, we do not have a space X for which $wee(X) < we(X^2)$.

Lemma 3.2. *If $wee(X) = \omega$ and F is a closed subset of X^2 and $\mathcal{U}$ is a cover of F by open subsets of X^2, then there is a countable subset A of X such that*

$$F \subseteq \mathrm{St}(X \times A, \mathcal{U}).$$

Theorem 3.3. *If X has a zero-set diagonal and $wee(X) = \omega$, then X is submetrizable.*

Proof. Let $f : X^2 \to [0,1]$ be such that $f^{-1}(0) = \Delta_X$. Next, for $n \in \mathbb{N}$ we let $C_n = f^{-1}([1/n, 1])$. Of course C_n is a closed subset of X^2, and $X^2 \setminus \Delta_X = \bigcup_{n \in \mathbb{N}} C_n$.

For $n \in \mathbb{N}$, we let $\mathcal{W}_n$ be defined by

$$\mathcal{W}_n = \{U \times V : U \times V \subseteq f^{-1}((1/2n, 1]),\ V \times V \subseteq f^{-1}([0, 1/2n))\ \&\ U, V \text{open in } X\}.$$

Note that $\mathcal{W}_n$ is a cover of C_n by open subsets of X^2. To see this, fix $n \in \mathbb{N}$ and let $(x, y) \in C_n$. We have $f(x, y) \in (1/2n, 1]$, and therefore there exist open subsets U and V of X such that $(x, y) \in U \times V \subseteq f^{-1}((1/2n, 1])$. Moreover, since $(y, y) \in V \times V$ and $f(y, y) = 0$ we can shrink V in such a way that $V \times V \subseteq f^{-1}([0, 1/2n))$.

Since $wee(X) = \omega$, by the preceding lemma we may find a countable subset B_n of X such that

$$C_n \subseteq \mathrm{St}(X \times B_n, \mathcal{W}_n).$$

We now let $B = \bigcup_{n \in \mathbb{N}} B_n$, and we define $F : X \to [0, 1]^B$ by

$$F(x)(b) = f(x, b).$$

We will show that F is an injection. Since B is countable, this will imply that X is submetrizable. Pick $x, y \in X$ with $x \neq y$. Then there is some $n \in \omega \setminus \{0\}$ with $(x, y) \in C_n$. So we may find $b \in B_n$ and $U \times V \in \mathcal{W}_n$ such that $(x, y) \in U \times V$ and $b \in V$. Then $(x, b) \in U \times V$ and $(y, b) \in V \times V$. From the definition of $\mathcal{W}_n$, it follows that

$$f(y, b) < 1/2n < f(x, b),$$

and therefore $F(x) \neq F(y)$. This completes the proof. $\square$

The following is the announced generalization of [13, Theorem 1] and [7, Theorem 2.1].

Corollary 3.4. *If X^2 has countable weak extent and a zero-set diagonal, then X is submetrizable.*

In [7, Theorem 2.4 and 2.5], R.Z. Buzyakova proved that if X has a regular G_δ-diagonal and either it is separable or X^2 has countable extent, then X condenses onto a second-countable Hausdorff space.

Following the same technique of Buzyakova, we now generalize those two results.

Theorem 3.5. *Let $wee(X) \leq \kappa$ and assume that X has a regular G_δ-diagonal. Then X condenses onto a Hausdorff space of weight at most κ.*

Proof. Let $\Delta_X = \bigcap_{n<\omega} U_n = \bigcap_{n<\omega} \overline{U}_n$, and let $C_n = X^2 \setminus U_n$. We define a family of open sets $\mathcal{U}$ as follows:

$$\mathcal{U} = \{U \times V : U \times V \subset X \setminus \overline{U}_m, V \times V \subset U_m \text{ for some } m \in \omega\ \&\ U, V \text{open in } X\}.$$

Note that since $\Delta_X = \bigcap_{m \in \omega} \overline{U}_m$, it follows that $\mathcal{U}$ is an open cover of $X^2 \setminus \Delta_X$.

Since $wee(X) \leq \kappa$, we may find, for every $n \in \omega$, a subset B_n of X of cardinality at most κ such that

$$C_n \subseteq \mathrm{St}(X \times B_n, \mathcal{U}).$$

If we let $B = \bigcup_{n \in \omega} B_n$, then B is of cardinality at most κ and

$$X^2 \setminus \Delta_X \subseteq \mathrm{St}(X \times B, \mathcal{U}).$$

Now we let the family $\mathcal{B}$ consist of all open subsets of X of one of the following forms:

1. $\{y : (y, b) \in U_n\}$ for some $b \in B$ and some $n \in \omega$,
2. $\{x : (x, b) \in X^2 \setminus \overline{U}_n\}$ for some $b \in B$ and some $n \in \omega$.

Then since $|B| \leq \kappa$, we also have that $|\mathcal{B}| \leq \kappa$. We will show that $\mathcal{B}$ is a Hausdorff separating family (cf. [7]).

So, pick $p \neq q$. Then there is some $b \in B$ and $U \times V \in \mathcal{U}$ such that $b \in V$ and $(p,q) \in U \times V$. Also, since $U \times V \in \mathcal{U}$, there is some $m \in \omega$ such that

$$U \times V \subset X \setminus \overline{U}_m \ \& \ V \times V \subset U_m.$$

This means that $(p,b) \in U_m$ and $(q,b) \in X \setminus \overline{U}_m$, and so we have

$$\begin{array}{rcl} p & \in & \{y : (y,b) \in U_m\} \\ q & \in & \{x : (x,b) \in X^2 \setminus \overline{U}_m\}, \end{array}$$

and since these open sets are disjoint members of $\mathcal{B}$, this shows that $\mathcal{B}$ is Hausdorff separating. □

Corollary 3.6. *If X^2 has countable weak extent and a regular G_δ-diagonal, then X condenses onto a second countable Hausdorff space.*

4. Some cardinal inequalities

In this section we prove various cardinality bounds involving different types of diagonal degree. We start off by showing that for Hausdorff spaces X the inequalities $|X| \leq 2^{d(X)s\Delta(X)}$ and $|X| \leq we(X)^{\Delta_2(X)}$ hold.

Next, we shall prove that if X is either a Baire space with a rank 2-diagonal or a space with a rank 3-diagonal, then its cardinality is bounded by $wL(X)^\omega$. We do not know if the same inequality is still true for spaces having a strong rank 2-diagonal. However, we can prove that, for such spaces, the inequality $|X| \leq wL(X)^{\pi\chi(X)}$ holds. Finally, we will show that the last formula is true for homogeneous spaces having a regular G_δ-diagonal.

Proposition 4.1. *For any Hausdorff space X we have*

$$|X| \leq 2^{d(X)s\Delta(X)}.$$

Proof. Let $\kappa = d(X)s\Delta(X)$ and fix a family $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ that witnesses the fact that X has a strong rank 1 G_κ-diagonal. Let D be a dense subset of X of cardinality at most κ. We define a map $F : X \to \mathcal{P}(D)^\kappa$ by

$$F(x)(\alpha) = D \cap \mathrm{St}(x,\mathcal{U}_\alpha).$$

We only have to show that this map is one-to-one. First of all, note that since D is dense, we always have $x \in \overline{F(x)(\alpha)}$. Now let $x \neq y$. Then we may find $\alpha < \kappa$ with $y \notin \overline{\mathrm{St}(x,\mathcal{U}_\alpha)}$. But then, since $F(x)(\alpha) \subseteq \mathrm{St}(x,\mathcal{U}_\alpha)$, it follows that $y \notin \overline{F(x)(\alpha)}$. So as $y \in \overline{F(y)(\alpha)}$, it follows that $F(x)(\alpha) \neq F(y)(\alpha)$. □

One could try to conjecture the bound $2^{d(X)\Delta(X)}$, but the Katetov extension of the discrete space ω disproves it. It is separable, it has a G_δ-diagonal and its cardinality is $2^{\mathfrak{c}}$.

Taking into account a result of Ginsburg and Woods, see [11, Theorem 9.4], which states that if X is a T_1 space, then its cardinality is bounded by $2^{e(X)\Delta(X)}$, it is quite natural to wonder whether the previous proposition can be improved as follows:

Question 4.2. *Is the cardinality of a Hausdorff space X bounded by $2^{we(X)s\Delta(X)}$?*

If, in the previous question, we replace $s\Delta(X)$ with $\Delta_2(X)$, we can actually prove the following stronger bound.

Proposition 4.3. *For any Hausdorff space X we have*

$$|X| \leq we(X)^{\Delta_2(X)}.$$

Proof. Let $\kappa = we(X)$ and $\lambda = \Delta_2(X)$. Fix a sequence of open covers $\{\mathcal{U}_\alpha : \alpha < \lambda\}$ witnessing the fact that X has a rank 2 G_λ-diagonal. For every $\alpha < \lambda$, we may fix a subset A_α of X with $|A_\alpha| \leq \kappa$ such that $X = \mathrm{St}(A_\alpha, \mathcal{U}_\alpha)$. We let $A = \bigcup_{\alpha<\lambda} A_\alpha$. Note that $|A| \leq \kappa \cdot \lambda$.

We may fix a map $f : X \to A^\lambda$ with the property that for $x \in X$ and $\alpha < \lambda$ we have that $f(x)(\alpha) = a \in A_\alpha$ and $x \in \mathrm{St}(a, \mathcal{U}_\alpha)$. To complete the proof we will show that such a mapping is injective.

So fix $x \neq y$. Then we may find $\alpha < \lambda$ such that

$$\mathrm{St}(x, \mathcal{U}_\alpha) \cap \mathrm{St}(y, \mathcal{U}_\alpha) = \emptyset.$$

Now let $p = f(x)(\alpha)$. Then $x \in \mathrm{St}(p, \mathcal{U}_\alpha)$, and so also $p \in \mathrm{St}(x, \mathcal{U}_\alpha)$. This means that $p \notin \mathrm{St}(y, \mathcal{U}_\alpha)$ and therefore $y \notin \mathrm{St}(p, \mathcal{U}_\alpha)$. This implies that $p \neq f(y)(\alpha)$. So the mapping f is injective and this completes the proof. □

This result should be compared with the inequality $|X| \leq we(X)^{psw(X)}$, obtained by R. Hodel (see [3] for an alternative and direct proof; see also [12]). The Katetov extension of ω witnesses that in the last two formulas it is not possible to put $\Delta(X)$ at the exponent. However, one may still try to conjecture to improve Ginsburg-Woods' inequality by moving down $e(X)$ from the exponent. This question was already published by Bella in 1996 (see [6]), but we think is worthy to repeat it here.

Question 4.4. *Does the inequality*

$$|X| \leq e(X)^{\Delta(X)}$$

hold for any T_1 space X?

In [4, Theorem 2], Bella proved that the cardinality of a Hausdorff space X is bounded by $2^{c(X)\Delta_2(X)}$. This was done by an application of the Erdös-Rado Theorem. For Baire spaces with a rank 2-diagonal this bound can be considerably improved.

Proposition 4.5. *If X a Baire space with a rank 2-diagonal then,*

$$|X| \leq wL(X)^\omega.$$

Proof. This follows from Proposition 4.3, the fact that $we(X) \leq d(X)$ and the following lemma. □

Lemma 4.6. *If X is a Baire space with a G_δ-diagonal then,*

$$d(X) \leq wL(X)^\omega.$$

Proof. Let $wL(X) = \kappa$ and let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers of X witnessing the fact that X has a rank 1-diagonal. For every $n < \omega$, we fix a family $\mathcal{V}_n \subseteq \mathcal{U}_n$ of cardinality κ whose union is dense in X. Next we let $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$ and $D_n = \bigcup \mathcal{V}_n$. Then $|\mathcal{V}| \leq \kappa$, and D_n is an open and dense subset of X for every n. Since X is a Baire space, this means that $D = \bigcap_{n<\omega} D_n$ is a dense subset of X. So to complete the proof it suffices to show that $|D| \leq \kappa^\omega$.

We fix some well-ordering on $\mathcal{V}$ and we define a map $f : D \to \mathcal{V}^\omega$ as follows

$$f(d)(n) = \min\{V \in \mathcal{V} : d \in V \in \mathcal{V}_n\}.$$

We will show that f is an injection. So fix $x, y \in D$ with $x \neq y$. Then $y \notin \mathrm{St}(x,\mathcal{U}_n)$ for some $n \in \omega$. Let $V = f(x)(n)$. Then $x \in V$ and since $\mathcal{V}_n$ is a refinement of $\mathcal{U}_n$, this means that $V \subseteq \mathrm{St}(x,\mathcal{U}_n)$. So we have that $y \notin V$ and therefore $f(x)(n) \neq f(y)(n)$. This completes the proof. □

We could ask whether the Baire assumption in Proposition 4.5 is necessary. This is an open question, but we can prove that for spaces having a rank 3-diagonal the following is true.

Proposition 4.7. *If X has a rank 3-diagonal then,*

$$|X| \leq wL(X)^\omega.$$

Proof. Let $wL(X) = \kappa$ and let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers of X witnessing the fact that X has a rank 3-diagonal. For every $n < \omega$, we fix a family $\mathcal{V}_n \subseteq \mathcal{U}_n$ of cardinality κ whose union is dense in X.

Next we let $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$. Of course we have $|\mathcal{V}| \leq wL(X)$. Note that whenever $U \in \mathcal{U}_n$, there is some $V \in \mathcal{V}_n$ such that $U \cap V \neq \emptyset$. So it follows that for every $x \in X$ and $n \in \omega$, there is some $V \in \mathcal{V}_n$ such that $\mathrm{St}(x,\ \mathcal{U}_n) \cap V \neq \emptyset$. Also note that in this case $V \subseteq \mathrm{St}^2(x,\ \mathcal{U}_n)$. We fix a well-ordering on $\mathcal{V}$ and we define a map $F : X \to \mathcal{V}^\omega$ as follows

$$F(x)(n) = \min\{V \in \mathcal{V} : V \in \mathcal{V}_n\ \&\ \mathrm{St}(x,\ \mathcal{U}_n) \cap V \neq \emptyset\}.$$

We have just shown that F is well-defined. It remains to show that F is an injection. So let $x, y \in X$ with $x \neq y$. By assumption, there is some $n \in \omega$ such that

$$\mathrm{St}^2(x,\ \mathcal{U}_n) \cap \mathrm{St}(y,\ \mathcal{U}_n) = \emptyset.$$

Since $F(x)(n) \subseteq \mathrm{St}^2(x,\ \mathcal{U}_n)$ and $F(y)(n) \cap \mathrm{St}(y,\ \mathcal{U}_n) \neq \emptyset$, it follows that $F(x)(n) \neq F(y)(n)$. This shows that F is an injection and this completes the proof. □

The discrete cellularity of a space X is the cardinal number $dc(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a discrete family of open subsets of $X\}$. The last result should be compared with the inequality $|X| \leq 2^{dc(X)\Delta_3(X)}$ proved in [5]. Note that, at least for regular spaces, we have $dc(X) \leq wL(X)$ and the gap can be artitrarely large. We do not know if the last two mentioned inequalities are true for spaces with a strong rank 2-diagonal.

Question 4.8. *Let X be a space with a strong rank 2-diagonal. Is it the case that*

- $|X| \leq wL(X)^\omega$ *?*
- $|X| \leq 2^{dc(X)}$ *?*

However, for spaces of countable π-character, we have the answer.

Proposition 4.9. *Let X be a space with a strong rank 2-diagonal. Then*

$$|X| \leq wL(X)^{\pi\chi(X)}.$$

Proof. Let $\{\mathcal{U}_n : n < \omega\}$ be a sequence of open covers of X witnessing the fact that X has a strong rank 2-diagonal and let $\kappa = \pi\chi(X)$ and $\lambda = wL(X)$. For every $x \in X$, we let $\mathcal{V}_x = \{V(x, \alpha) : \alpha < \kappa\}$ be a local π-base at x. For $n < \omega$, we fix a family $\mathcal{W}_n \subseteq \mathcal{U}_n$ of cardinality λ whose union is dense in X.

Next we let $\mathcal{W} = \bigcup_{n<\omega} \mathcal{W}_n$. Note that $|\mathcal{W}| \leq \lambda$. Since $\mathcal{U}_n$ is a cover of X, it follows that whenever V is a non-empty open subset of X, then $V \cap W \neq \emptyset$ for some $W \in \mathcal{W}_n$. We fix a well-ordering on $\mathcal{W}$ and we define a map $F : X \to \mathcal{W}^{\kappa\times\omega}$ as follows,

$$F(x)(\alpha, n) = \begin{cases} \emptyset, & \text{if } V(x,\alpha) \not\subseteq \mathrm{St}(x,\ \mathcal{U}_n), \\ \min\{W \in \mathcal{W}_n : W \cap V(x,\alpha) \neq \emptyset\}, & \text{otherwise.} \end{cases}$$

By the remarks made before, the map F is well-defined. For $x \in X$ and $n < \omega$, we let $W(x, n)$ be defined by

$$W(x,n) = \bigcup\{F(x)(\alpha, n) : \alpha \in \kappa\}.$$

Note that by definition of F, we have that $W(x, n) \subseteq \mathrm{St}(\mathrm{St}(x,\ \mathcal{U}_n),\ \mathcal{W}_n)$ and since $\mathcal{W}_n$ is a refinement of $\mathcal{U}_n$, it follows that

$$W(x,n) \subseteq \mathrm{St}^2(x,\ \mathcal{U}_n).$$

Claim. $x \in \overline{W(x,n)}$ for every $n \in \omega$.

Proof of Claim. To see this, let Ox be an open neighbourhood of x. Then $V(x,\alpha) \subseteq Ox \cap \mathrm{St}(x,\ \mathcal{U}_n)$ for some $\alpha < \kappa$. By definition of F, it follows that $F(x)(\alpha, n) \cap V(x,\alpha) \neq \emptyset$ and therefore $F(x)(\alpha, n) \cap Ox \neq \emptyset$. Since $F(x)(\alpha, n) \subseteq W(x,n)$, it follows that $x \in \overline{W(x,n)}$ and this proves the claim. ◀

So for every $x \in X$, we have that

$$\{x\} \subseteq \bigcap_{n<\omega} \overline{W(x,n)} \subseteq \bigcap_{n<\omega} \overline{\mathrm{St}^2(x,\ \mathcal{U}_n)} = \{x\}.$$

This shows that F is an injection and this completes the proof. □

For homogeneous spaces, the previous proposition can be improved.

Note that if X is homogeneous and $\pi\chi(X) = \kappa$, then there is a collection $\{V(x,\alpha) : x \in X, \alpha < \kappa\}$ of non-empty open subsets of X such that for every $x \in X$, $\mathcal{V}_x = \{V(x,\alpha) : \alpha < \kappa\}$ is a local π-base at x and whenever Ox and Oy are open neighbourhoods of x and y respectively, there is some $\alpha < \kappa$ such that

$$V(x,\alpha) \subseteq Ox \text{ and } V(y,\alpha) \subseteq Oy.$$

For example, if $p \in X$ is fixed and $\{V_\alpha : \alpha < \kappa\}$ is a local π-base at p in X, then we may define $V(x,\alpha) = h_x[V_\alpha]$, where h_x is a homeomorphism of X mapping p onto x.

Proposition 4.10. *Let X be a homogeneous space with a regular G_δ-diagonal. Then*

$$|X| \leq wL(X)^{\pi\chi(X)}.$$

Proof. Fix a sequence $\{\mathcal{U}_n : n < \omega\}$ of open covers of X witnessing the fact that X has a regular G_δ-diagonal. Furthermore, let $\pi\chi(X) = \kappa$ and $wL(X) = \lambda$ and fix a collection $\{V(x,\alpha) : x \in X, \alpha < \kappa\}$ of non-empty open subsets of X with the property stated just before this proposition.

Next, for $n < \omega$, we fix a family $\mathcal{W}_n \subseteq \mathcal{U}_n$ of cardinality λ whose union is dense in X.

Note that since $\mathcal{U}_n$ is a cover of X, if follows that whenever V is a non-empty open subset of X, then $V \cap W \neq \emptyset$ for some $W \in \mathcal{W}_n$. We let $\mathcal{W} = \bigcup_{n<\omega} \mathcal{W}_n$ and we fix a well-ordering on $\mathcal{W}$. Note that $|\mathcal{W}| \leq wL(X)$.

We now define a map $F : X \to \mathcal{W}^{\omega\times\kappa}$ as follows,

$$F(x)(n,\alpha) = \min\{W \in \mathcal{W} : W \in \mathcal{W}_n \ \&\ W \cap V(x,\alpha) \neq \emptyset\}.$$

We have just showed that F is well-defined. It remains to verify that F is an injection, so let $x, y \in X$ with $x \neq y$. Then there is some $n < \omega$ and open neighbourhoods Ox and Oy of x and y respectively such that

$$\mathrm{St}(Ox,\ \mathcal{U}_n) \cap Oy = \emptyset.$$

By the property of our local π-bases, it follows that there is some $\alpha < \kappa$ such that

$$V(x,\alpha) \subseteq Ox \text{ and } V(y,\alpha) \subseteq Oy.$$

Now recall that $\mathcal{W}_n$ is a refinement of $\mathcal{U}_n$, and therefore, since $V(x,\alpha) \subseteq Ox$, we have the following:

$$F(x)(n,\alpha) \subseteq \mathrm{St}(Ox,\ \mathcal{U}_n).$$

Furthermore, by construction we have that $F(y)(n,\alpha) \cap Oy \neq \emptyset$ so it follows that $F(x)(n,\alpha) \neq F(y)(n,\alpha)$. This shows that F is an injection and this completes the proof. □

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